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DENSITY THEOREMS FOR FINITISTIC TREES

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This paper investigates to what extent, the Milliken partition theorem for finitistic trees is a density result.

1. Introduction

In recent years several results appeared showing that certain partition (Ramsey-type) theorems are rather density phenomena than merely partition theorems. Let us mention e.g. Szemerédi's result on arithmetic progressions [8], Fürstenberg's and Katznelson's generalization of it [3], the geometric density theorem of Brown and Buhler [1] and Rödl's result on points in power-set lattices [6] (generalizing former results of Erdös and Kleitman [2] and Sperner [7]). Of course not every partition theorem admits a density version, e.g. Turán's extremal theorem on triangle free graphs shows that all nontrivial cases of Ramsey's theorem have no generalization to a density result.

In this note we investigate to what extent density results for finitistic trees may be obtained. Surprisingly it turns out that although the general case is not a density phenomenon, certain special cases can be obtained as density results.

2. Results

A *tree* is a partially ordered set $(T, \leqq)$ such that for every element $x \in T$ the set $\{y \in T | y < x\}$ of predecessors of x is totally ordered. The cardinality $|\{y \in T | y < x\}|$ is called the *rank of* x, for nonnegative integers k we denote by $T(k)$ the kth level of T, i.e. $T(k) = \{x \in T | \text{rank}(x) = k\}$. A *finitistic tree* is a tree with a least element, which is called the *root of* T, such that all elements have finite rank and such that each level is a finite set. Obviously for each element x in a finitistic tree the set of immediate successors is a finite set, the cardinality of this set is called the *degree of* x. A finitistic tree is *regular* if all elements have the same degree.

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A subset $\hat{T} \subseteqq T$ is a *strongly embedded subtree* (with respect to the order induced from T) iff

(1) all infima in $\hat{T}$, resp. in T coincide, i.e. $\inf_{\hat{T}}(x, y) = \inf_T(x, y)$ for all $x, y \in \hat{T}$, viz. $\hat{T}$ is infimum preserving,

(2) all degrees in $\hat{T}$, resp. in T coincide, i.e. $\deg_{\hat{T}}(x) = \deg_T(x)$ for every $x \in \hat{T}$, viz. $\hat{T}$ is degree preserving,

(3) $\hat{T}$ is level preserving, i.e. $\operatorname{rank}_{\hat{T}}(x) = \operatorname{rank}_{\hat{T}}(y)$ iff $\operatorname{rank}_T(x) = \operatorname{rank}_T(y)$ for all $x, y \in \hat{T}$.

Condition 3 implies that for each strongly embedded subtree $\hat{T}$ of T there can be defined a strictly ascending mapping $f: \omega \to \omega$, viz. the *level-assignment function*, by letting $f(k) = l$ iff $\operatorname{rank}_T(x) = l$ for every $x \in \hat{T}$ with $\operatorname{rank}_{\hat{T}}(x) = k$.

As usual, ω (the first infinite ordinal) denotes the set of all nonnegative integers. For infinite subsets $A \subseteqq \omega$ we denote by $[A]^\omega$ the set of all infinite subsets of A. Now the simplest case of the Laver—Pincus—Milliken version of the Halpern—Läuchli partition theorem says:

Theorem 2.1. ([4], [5]) *Let T be a finitistic tree and let r be a positive integer. Then for every coloring $\Delta: T \to r$ there exists a strongly embedded subtree $\hat{T} \subseteqq T$ such that the restriction $\Delta \upharpoonright \hat{T}$ is a constant coloring.*

Here we prove a density version of this result for regular finitistic trees and show that in some sense this is best possible.

Definition 2.2. Let T be a finitistic tree. A subset $S \subseteqq T$ has *positive upper density* iff

$$\limsup_{k \to \infty} \frac{|T(k) \cap S|}{|T(k)|} > 0.$$

Particularly for totally ordered trees (i.e. $T = \omega$) positive upper density simply means that S is infinite, thus this notion of positive upper density is considerably weaker than requiring e.g.

$$\limsup_{n \to \infty} \frac{\left| \bigcup_{k<n} T(k) \cap S \right|}{\left| \bigcup_{k<n} T(k) \right|} > 0.$$

Theorem 2.3. *Let T be a regular finitistic tree and let $S \subseteqq T$ be a set of positive upper density. Then there exists a strongly embedded subtree $\hat{T} \subseteqq T$ which is contained in S, i.e. $\hat{T} \subseteqq S$.*

The following counterexamples show that in a sense regularity is a necessary condition for theorem 2.3:

Theorem 2.4. *Let $\varepsilon > 0$ be a real number. Then there exists a finitistic tree T such that every element of T has degree one or two and there exists a subset $S \subseteqq T$ such that*

$$\frac{|T(k) \cap S|}{|T(k)|} > 1 - \varepsilon \quad \textit{for every} \quad k \geqq 0,$$

but there does not exist any strongly embedded subtree $\hat{T}\subseteq T$ which is contained in S, i.e. such that $\hat{T}\subseteq S$.

Theorem 2.5. *Let $\varepsilon>0$ be a real number. Then there exists a finitistic tree T such that T is semi-regular in the sense that every two elements of same rank always have same degree and there exists a subset $S\subseteq T$ such that*

$$\frac{|T(k)\cap S|}{|T(k)|} > 1-\varepsilon \quad \textit{for every} \quad k \geqq 0,$$

but there does not exist any strongly embedded subtree $\hat{T}\subseteq T$ which is contained in S, i.e. such that $\hat{T}\subseteq S$.

3. Proof of the positive result

For the remainder of this section let T denote a regular finitistic tree. For elements $x\in T$ let $I(x)=\{y\in T|x\leqq y\}$ be the set of successors of x. Analogously $I(X)=\bigcup\{I(x)|x\in X\}$ for subsets $X\subseteq T$. For any subset $S\subseteq T$ and nonnegative integer n, let $S(n)=S\cap T(n)$.

Lemma 3.1. *Let T be a regular finitistic tree, let $m<n$ be nonnegative integers and let $x, y\in T(m)$ be elements of rank m, then $|I(x)\cap T(n)|=|I(y)\cap T(n)|=d^{n-m}$, where d is the common degree of the elements in T.*

Proof. Obvious. ∎

Lemma 3.2. *Let $S\subseteq T$ be a subset of positive upper density. Then there exists an $\varepsilon>0$ and an infinite subset $W\in[\omega]^\omega$ such that*

$$|S(n)| > \varepsilon\, d^n \quad \textit{for every} \quad n\in W.$$

Proof. Obvious from definition 2.2.

Lemma 3.3. *Let $m<n$ be nonnegative integers and let $S(m)\subseteq T(m)$ such that*

$$|S(m)| > \varepsilon d^m.$$

Then

$$\left|I(S(m))\cap T(n)\right| > \varepsilon d^n.$$

Proof. Obvious from regularity. ∎

Lemma 3.4. *Let $S\subseteq T$, $W\in[\omega]^\omega$ and let $\varepsilon>0$. Then there exists a positive integer $m=m(\varepsilon, W, S)$ such that*

$$\left|S(n)\setminus\bigcup\{I(S(k))|k \leqq m\}\right| \leqq \varepsilon\, d^n$$

for every $n\in W$ with $m<n$.

Proof. Assume to the contrary that for every m there exists an $n\in W$ such that

$$\left|S(n)\setminus\bigcup\{I(S(k))|k \leqq m\}\right| > \varepsilon\, d^n$$

Then there exists a strictly increasing sequence $m_0 < m_1 < \dots$ of positive integers in W such that

$$\left|S(m_{i+1}) \setminus \bigcup \{I(S(k)) | k \leqq m_i\}\right| > \varepsilon d^{m_{i+1}}$$

for every $i < \omega$. By lemma 3.3. this leads eventually to a contradiction, viz. consider any $i > (\varepsilon)^{-1}$. ▮

Lemma 3.5. *Let $S \subseteqq T$, $W \in [\omega]^\omega$ and let $\varepsilon, \delta > 0$ be such that*

$$|S(n)| > (\varepsilon + \delta)\, d^n \quad \textit{for every} \quad n \in W.$$

Then there exists a positive integer $l \leqq m(\delta, W, S)$, an element $a \in S(l)$ and an infinite set $W^* \in [W]^\omega$ such that

$$|I(a) \cap S(n)| > \varepsilon\, d^{n-l} \quad \textit{for every} \quad n \in W^*.$$

Proof. Let $A \subseteqq \bigcup \{S(k) | k \leqq m(\delta, W, S)\}$ be a maximal antichain. Then

$$|I(A) \cap S(n)| = \sum_{a \in A} |I(a) \cap S(n)| \quad \text{for every} \quad n > m(\delta, W, S)$$

as any two different elements of A are incomparable and $T(n) \cap I(A) = \bigcup \{I(S(k)) \cap \cap T(n) | k \leqq m(\delta, W, S)\}$ for every $n > m(\delta, W, S)$ because A is maximal.

By the pigeon-hole principle there exists an infinite subset $W^* \in [W]^\omega$ and an element $a \in A$ satisfying the requirements of the lemma. Otherwise it would follow that

$$|I(A) \cap S(n)| \leqq \varepsilon\, d^n$$

for infinitely many $n \in W$, contradicting the choice of $m(\delta, W, S)$. ▮

Corollary 3.6. *Let $S \subseteqq T$ be a set of positive upper density. Then there exists a strongly embedded subtree $\hat{T} \subseteqq T$ such that*

(1) root $\hat{T} \in S$ *and*

(2) *$\hat{S} = S \cap \hat{T}$ has positive upper density with respect to $\hat{T}$, i.e.*

$$\limsup_{n \to \infty} \frac{|\hat{T}(n) \cap \hat{S}|}{|\hat{T}(n)|} > 0.$$

The next lemma provides the tools for constructing strongly embedded subtrees recursively:

Lemma 3.7. *Let $S \subseteqq T$ be with* root $T \in S$, *$W \in [\omega]^\omega$ and let $\varepsilon > 0$ be such that*

$$|S(n)| > \varepsilon\, d^n \quad \textit{for every} \quad n \in W.$$

Then there exists a nonnegative integer l, an element $a \in S(l)$ and an infinite subset $W^* \in [W]^\omega$ such that

$$|I(x) \cap S(n)| > (\varepsilon/4)\, d^{n-l-1}$$

for every $n \in W^*$ and every $x \in I(a) \cap T(l+1)$ (i.e. for every immediate successor x of a).

Proof. Assume to the contrary that the assertion is false. We construct recursively a sequence $(l_i, z_i, Y_i)_{i<\omega}$ of nonnegative integers l_i, elements $z_i \in S(l_i)$ and infinite subsets $Y_i \in [W]^\omega$ such that

$$|I(z_i) \cap S(n)| > \left(\frac{d-(1/2)}{d-1}\right)^i \varepsilon\, d^{n-l_i}$$

for every $i<\omega$ and $n \in Y_i$. Eventually this leads to a contradiction, viz. consider any $i > -(\log \varepsilon)\left(\log \frac{d-(1/2)}{d-1}\right)^{-1}$, thus proving the lemma.

Put $l_0=0$, $z_0=\text{root } T$ and $Y_0=W$. Assume that (l_i, z_i, Y_i) already has been constructed. As the assertion of the lemma is supposed to be false, there exists an element $x \in I(z_i) \cap T(l_i+1)$ and an infinite subset $Y \in [Y_i]^\omega$ such that

$$|I(x) \cap S(n)| \leqq (\varepsilon/4)\, d^{n-l_i-1} \quad \textit{for every} \quad n \in Y.$$

Then

$$\begin{aligned} |I(z_i) \cap S(n) \setminus I(x)| &> \left(\frac{d-(1/2)}{d-1}\right)^i \varepsilon\, d^{n-l_i} - \frac{\varepsilon}{4}\, d^{n-l_i-1} \\ &= \left(\frac{d-(1/2)}{d-1}\right)^i \varepsilon \left(d - \frac{1}{4}\left(\frac{d-1}{d-(1/2)}\right)^i\right) d^{n-l_i-1} \\ &\geqq \left(\left(\frac{d-(1/2)}{d-1}\right)^i \varepsilon \left(d-\frac{1}{2}\right) + \frac{\varepsilon}{4}\right) d^{n-l_i-1} \end{aligned}$$

for every $n \in Y$.

Thus there exists an element $z \in I(z_i) \cap T(l_i+1)$ and an infinite subset $Y' \in [Y]^\omega$ such that

$$|I(z) \cap S(n)| > \left(\left(\frac{d-(1/2)}{d-1}\right)^{i+1} \varepsilon + \frac{\varepsilon}{4(d-1)}\right) d^{n-l_i-1}$$

for every $n \in Y'$. Finally by lemma 3.5 there exists an $l_{i+1} > l_i$, an element $z_{i+1} \in I(z) \cap S(l_{i+1})$ and an infinite subset $Y_{i+1} \in [Y']^\omega$ such that

$$|I(z_{i+1}) \cap S(n)| > \left(\frac{d-(1/2)}{d-1}\right)^{i+1} \varepsilon\, d^{n-l_{i+1}}$$

for every $n \in Y_{i+1}$. ∎

Corollary 3.8. *Let* $S \subseteqq T$, $W \in [\omega]^\omega$ *and let* $\varepsilon > 0$ *be such that* $|S(n)| > \varepsilon d^n$ *for every* $n \in W$. *Then there exists an infinite subset* $W^* \in [W]^\omega$ *and for every* $l \in W^*$ *there exists an element* $z_l \in S(l)$ *and an* $\varepsilon_l > 0$ *such that* $|I(x) \cap S(n)| > \varepsilon_l d^{n-l-1}$ *for every* $n \in W^*$ *with* $n > l$ *and for every* $x \in I(z_l) \cap T(l+1)$.

Proof. By corollary 3.6 we can assume that root $T \in S$. Put $Y_0 = W$, $l_{-1} = -1$ and $\varepsilon_{-1} = \varepsilon$.

Assume that for some nonnegative integer i the set $Y_i \in [W]^\omega$, integers l_ν, elements z_{l_ν} and reals $\varepsilon_{l_\nu} > 0$ for $\nu < i$ have been defined in such a way that

(1) $l_0 < l_1 < \ldots < l_{i-1} < \min Y_i$

(2) $z_{l_\nu} \in S(l_\nu)$

(3) $|I(x) \cap S(n)| > \varepsilon_{l_\nu} d^{n-l_\nu-1}$ for every $x \in I(z_{l_\nu}) \cap T(l_\nu+1)$ and for every $n \in \{l_\mu | \nu < \mu < i\} \cup Y_i$.

Let l_i, $z_{l_i} \in S(l_i)$, $\varepsilon_{l_i} > 0$ and $Y_{i+1} \in [Y_i]^\omega$ be according to lemma 3.7 for $\varepsilon_{l_{i-1}}$, Y_i and S. Obviously properties (1), (2) and (3) are satisfied again. Finally $W^* = \{l_i | i < \omega\}$ has the desired properties. ∎

Now corollary 3.8 is used in order to construct a strongly embedded subtree which is contained in S recursively.

Proof of Theorem 2.3. Let $S \subseteq T$ be a set of positive upper density. According to lemma 3.2 and lemma 3.7 there exists a nonnegative integer l, an element $z \in S(l)$, an infinite subset $W \in [\omega]^\omega$ and an $\varepsilon > 0$ such that

$$|I(x) \cap S(n)| > \varepsilon\, d^{n-l-1}$$

for every $n \in W$ and every $x \in I(z) \cap T(l+1)$. The element z will serve as the root of the strongly embedded subtree $\hat{T}$ that we are going to construct.

By induction we can assume that $\bigcup \{\hat{T}(k) | k < m\}$ has been constructed satisfying the following additional properties: Say that $\hat{T}(m-1) = \{z_\mu | \mu < d^{m-1}\} \subseteq T(l)$, viz. the $(m-1)$-st level of $\hat{T}$ is embedded into the lth level of T, there exists an infinite subset $W \in [\omega]^\omega$ and a real number $\varepsilon > 0$ such that $|I(x) \cap S(n)| > \varepsilon d^{n-l-1}$ for every $n \in W$, every $\mu < d^{m-1}$ and $x \in I(z_\mu) \cap T(l+1)$.

Apply corollary 3.8 to each $I(x) \cap S$ in order to obtain an infinite subset $W^* \in [\omega]^\omega$, an $\varepsilon^* > 0$, an $l^* > l$ and for each $x \in \bigcup \{I(z_\mu) \cap T(l+1) | \mu < d^{m-1}\}$ an element $z(x) \in I(x) \cap S(l^*)$ such that

$$|I(y) \cap S(n)| > \varepsilon^*\, d^{n-l^*-1}$$

for every $y \in I(z(x)) \cap T(l^*+1)$, $x \in \bigcup \{I(z_\mu) \cap T(l+1) | \mu < d^{m-1}\}$ and $n \in W^*$.

These elements $z(x)$ then can serve as the mth level of $\hat{T}$. ∎

4. Counterexamples

Proof of Theorem 2.4. Consider the binary tree $2^{<\omega}$, viz. elements are finite $0-1$ sequences and $(\alpha_0, \ldots, \alpha_{m-1}) \leqq (\beta_0, \ldots, \beta_{n-1})$ iff $(\alpha_0, \ldots, \alpha_{m-1})$ is an initial segment of $(\beta_0, \ldots, \beta_{n-1})$, i.e. $m \leqq n$ and $\alpha_\nu = \beta_\nu$ for every $\nu < m$.

For $\varepsilon > 0$ let the sequence $(n_i)_{i<\omega}$ be defined as follows:

(i) $n_0 = 1$

(ii) n_{i+1} is the minimal integer larger than n_i satisfying

$$\varepsilon \left(2^{n_{i+1}-i-1} + \sum_{\nu=0}^{i} 2^{n_\nu - \nu - 1} \right) > \sum_{\nu=0}^{i} 2^{n_\nu - \nu - 1}.$$

For a sequence $(\alpha_0, \ldots, \alpha_{m-1}) \in 2^{<\omega}$, we denote by $\mu(\alpha_0, \ldots, \alpha_{m-1})$ the minimal index i such that $\alpha_i = 0$, let $\mu(\alpha_0, \ldots, \alpha_{m-1}) = \infty$ if no such i exists. Let $T = \{(\alpha_0, \ldots, \alpha_{m-1}) \in 2^{<\omega} \mid \text{for every index } m > j > n_{\mu(\alpha_0, \ldots, \alpha_{m-1})} \text{ it follows that } \alpha_j = 0\}$, i.e. basically T contains all sequences $a(i) = (\underbrace{1, \ldots, 1}_{i}, 0)$ and their predecessors,

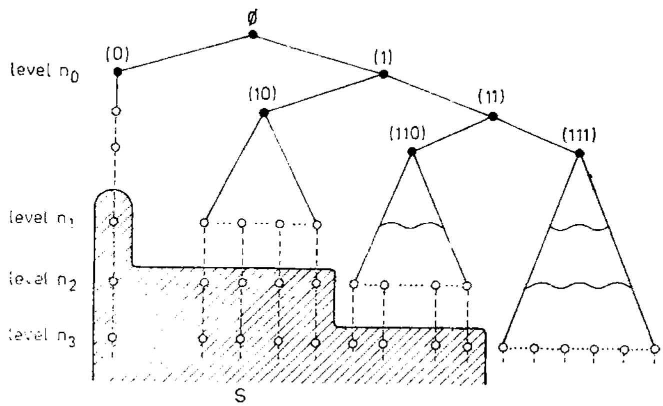

Fig. 1

the successors of $a(i)$ in T are branching twice until level n_i is reached, then they are prolonged by chains. Consider also Figure 1.

Let $S=\{(\alpha_0, \ldots, \alpha_{m-1})\in T \mid m \geqq n_{1+\mu(\alpha_0, \ldots, \alpha_{m-1})}\}$. Then $\frac{|T(k)\cap S|}{|T(k)|}<\varepsilon$ for every $k\geqq 0$ by property (ii) of the sequence $(n_i)_{i<\omega}$. Note that S is upwards closed, thus its complement $T\setminus S$ has density at least $(1-\varepsilon)$ and contains precisely one infinite chain, viz. $\emptyset$, (1), $(1, 1)$, $(1, 1, 1)$, ... Particularly $T\setminus S$ does not contain any strongly embedded subtree of T. ∎

Proof of Theorem 2.5. For $\varepsilon>0$ let the sequence $(n_i)_{i<\omega}$ satisfy $\sum_{i=0}^{\infty}\frac{1}{n_i}<\varepsilon$, where the n_i's are positive integers. Let T be the semi-regular finitistic tree consisting of all finite integer sequences $(\alpha_0, \ldots, \alpha_{m-1})$ such that $0\leqq\alpha_i<n_i$ for every $i<\omega$, viz. $T=\bigcup_{k<\omega}\prod_{j<k} n_i$. As before $(\alpha_0, \ldots, \alpha_{m-1})\leqq(\beta_0, \ldots, \beta_{n-1})$ iff $(\alpha_0, \ldots, \alpha_{m-1})$ is an initial segment of $(\beta_0, \ldots, \beta_{m-1})$. Let $S=\{(\alpha_0, \ldots, \alpha_{m-1})\in T \mid$ there exists an index $i<m$ such that $\alpha_i=0\}$. Then

$$\frac{|T(k)\cap S|}{|T(k)|}=\frac{\sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1}(n_j-1)\cdot\prod_{l=i+1}^{k-1}n_l\right)}{\prod_{i=0}^{k-1}n_i}\leqq\sum_{i=0}^{k-1}\frac{1}{n_i}<\varepsilon$$

for every $k\geqq 0$. Obviously then $T\setminus S$ has density at least $1-\varepsilon$ and does not contain any strongly embedded subtree of T. ∎

5. Concluding remarks

Using a compactness argument, e.g. König's lemma, yields a finite version of Theorem 2.3. Let us denote by T_d^n the tree consisting of the first n levels of the finitistic regular tree where all elements have degree d.

Theorem 5.1. *Let $\varepsilon>0$ be a real number and let d, m be positive integers. Then there exists a positive integer n such that for every subset $S \subseteq T_d^n$ with*

$$\frac{|T_d^n(k) \cap S|}{|T_d^n(k)|} > \varepsilon \quad \text{for every} \quad k < n$$

there exists a strongly embedded T_d^m-subtree which is contained in S.

The general version of the Halpern—Läuchli partition theorem says:

Theorem 5.2. ([4], [5]) *Let $(T_i)_{i<q}$ be a finite sequence of finitistic trees and let*

$$\Delta : \bigcup_{k<\omega} \prod_{i<q} T_i(k) \to r$$

be an r-coloring of the product of the levels of these trees, where q and r are positive integers. Then there exist strongly embedded subtrees $\hat{T}_i \subseteq T_i$, $i<q$, which all have the same level-assignment function, such that the restriction

$$\Delta \urcorner \bigcup_{k<\omega} \prod_{i<q} \hat{T}_i(k)$$

is a constant coloring.

Recall that Theorem 2.3 is a density version of Theorem 5.2 for the particular case where $q=1$ and T is regular, moreover the examples 2.4 and 2.5 show that regularity is a somewhat necessary assumption. We could not prove a density version of Theorem 5.2 for larger q's, but we would like to state this as a conjecture, viz.

Conjecture 5.3. Let $(T_i)_{i<q}$ be a sequence of finitistic regular trees and let

$$S \subseteq \bigcup_{k<\omega} \prod_{i<q} T_i(k)$$

be a set of positive upper density, i.e.

$$\limsup_{k\to\infty} \frac{\left|\left(\prod_{i<q} T_i(k)\right) \cap S\right|}{\left|\prod_{i<q} T_i(k)\right|} > 0,$$

then there exist strongly embedded subtrees $\hat{T}_i \subseteq T_i$, $i<q$, which all have the same level-assignment function, such that

$$\bigcup_{k<\omega} \prod_{i<q} \hat{T}_i(k) \subseteq S.$$

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